International Journal of Theoretical Physics, Vol. 40, No. 6, 2001

Precanonical Quantum Gravity: Quantization Without the Space–Time Decomposition

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Received August 27, 2000

A nonpertubative approach to quantum gravity using *precanonical* field quantization originating from the covariant De Donder–Weyl Hamiltonian formulation, which treats space and time variables on an equal footing, is presented. A generally covariant "multitemporal" generalized Schrödinger equation on the finite dimensional space of metric and space–time variables is obtained. An important ingredient of the formulation is the "bootstrap condition" which introduces a classical space–time geometry as an approximate concept emerging as the quantum average self-consistent with the underlying quantum dynamics. This ensures the independence of the theory from an arbitrarily fixed background. The prospects and unsolved problems of precanonical quantization of gravity are outlined.

1. INTRODUCTION

The goal of contemporary efforts in developing the quantum theory of gravity (for a recent review see, e.g., Isham, 1993, 1997; Rovelli, 1998) is to complete the synthesis of quantum theory and general relativity. This could be achieved either by developing a new "quantum–general-relativistic" framework in physics or by incorporating general relativity into a unifying quantum theory of all interactions. Two aspects of classical general relativity, which is at the same time the theory of space–time and the theory of the gravitational interaction, are being stressed then. Accordingly, quantization of gravity can be viewed either pragmatically, as a construction of the quantum field theory of gravity (see, e.g., Donoghue, 1994, 1996; Reuter, 1998, 2000) or, more conceptually, as a construction of the

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0020-7748/01/0600-1121$19.50/0

quantum theory of space–time. Both aspects are of course intimately related to each other.

By its very nature the program of *quantization* of gravity is an attempt to apply the principles of quantum theory, as we understand them at the present, to general relativity. In fact, it is (necessarily) understood even in a more narrow sense of imposing the presently known form of quantization rules, which proved to be successful for other fields evolving on a fixed space–time background, to general relativity with its diffeomorphism covariance and a dynamical space–time geometry. It is usually assumed that the currently practiced form of quantum (field) theory is applicable to general relativity without substantial modifications (cf. “generalized quantum mechanics” of Hartle (1994, 1995), which is constructed to be better suited to the context of quantum gravity).

However, it is well-known that the existing attempts to quantize gravity are confronted by both the problematic mathematical meaning of the involved constructions and the conceptual questions originating in difficulties of reconciling the fundamental principles of quantum theory with those of general relativity (see, e.g., Isham, 1993, 1997, for a review and further references). In particular, a distinct role of the time dimension in the probabilistic interpretation of quantum theory and in the formulation of quantum evolution laws seems to be contradistinguished from the equal rights status of space–time dimensions in the theory of relativity. The quintessential manifestation of this type of difficulties is known, somewhat loosely, as “the problem of time” (see, e.g., Isham, 1993, 1997; Kuchař, 1992; Unruh, 1993, for a review). Besides, the commonly adopted procedure of *canonical quantization* is preceded by the Hamiltonian formulation which requires a singling out of a time parameter and seems to be too tied to the classically inspired idea of evolution in time from a given Cauchy data. Technically, this procedure implies a global hyperbolicity of space–time, which seems to be a rather unnatural topological restriction for the expected quantum fluctuating “space–time foam” of quantum gravity.

The difficulties mentioned earlier are likely to indicate that the applicability of the conventional Hamiltonian methods in quantum theory of gravity can be rather limited. However, those difficulties could be partially overcome, or at least seen from another perspective, if we would have in our disposal a quantization procedure in field theory not so sensibly depending on space–time decomposition, that is, on the singling out of a time parameter.

One could argue that the path integral approach already embodies the idea. However, the particular path integral ansatz of Hawking’s Euclidean quantum gravity (Hawking, 1979) is in fact merely a symbolic solution to the Wheeler–DeWitt equation the derivation of which *is* substantially based on space–time decomposition. Moreover, the interpretation of this ansatz refers to spatial 3-geometries (in four dimensions). Besides, the usual path integral expression of the generating functional in field theory incorporates only the *time ordered* Green functions; hence, an implicit reference to a distinct time parameter.

In fact, what we need instead is a version of canonical quantization without a distinct role of time dimension and, therefore, one that is independent of the representation of fields as infinite dimensional systems evolving in time from the initial Cauchy data given on a space-like hypersurface. Clearly, such a "timeless" procedure of quantization, if exists, should have to be based on an analogue of Hamiltonian formalism in which space and time dimensions are treated on an equal footing.

Fortunately, although this seems to be not commonly known in theoretical physics, the Hamiltonian-like formulations of the field equations, which could be appropriate for a "timeless" version of canonical formalism, have been known in the calculus of variations already at least since the 30s. In the simplest version of those formulations, the so-called De Donder–Weyl (DW) theory (De Donder, 1935; Dickey, 1991, 1994; Kastrup, 1983; Rund, 1966; Weyl, 1935) the Euler–Lagrange field equations assume the following form of *DW Hamiltonian equations*

$$\partial_\mu y^a = \frac{\partial H}{\partial p^\mu_a}, \qquad \partial_\mu p^\mu_a = -\frac{\partial H}{\partial y^a}, \tag{1.1}$$

where y^a denote field variables, $p^\mu_a := \partial L/\partial(\partial_\mu y^a)$ are what we call *polymomenta*, $H := \partial_\mu y^a p^\mu_a - L$ is a function of (y^a, p^μ_a, x^ν) called the *DW Hamiltonian function*, and $L = L(y^a, \partial_\mu y^a, x^\nu)$ is a Lagrangian density.

Obviously, this form of the field equations can be viewed as a "multitemporal" or multiparameter generalization of Hamilton's canonical equations from mechanics to field theory in which the analogue of the configuration space is a finite dimensional space of field and space–time variables (y^a, x^ν) and the analogue of the extended phase space is a finite dimensional (extended) *polymomentum phase space* of the variables (y^a, p^μ_a, x^ν). In (1.1), fields are described essentially as a sort of multiparameter generalized (DW) Hamiltonian systems rather than as infinite dimensional mechanical systems, as in the standard Hamiltonian formalism. In doing so, the DW Hamiltonian function H, which thus far does not appear to have any evident physical interpretation, in a sense controls the space–time *variations* of fields, as specified by Eqs. (1.1), rather than their time *evolution*. The latter, however, is implicit in (1.1) in the case of hyperbolic field equations for which the Cauchy problem can be posed.

An intriguing feature of the framework under consideration is that in spite of the finite dimensionality of the polymomentum phase space it is capable to embrace the dynamics of fields, which usually are viewed as infinite dimensional Hamiltonian systems. From the equivalence of (1.1) to the Euler–Lagrange field equations, which is only restricted by the regularity of the DW Legendre transform $y^a_\nu \to p^\nu_a, L \to H$, it is obvious that no field degrees of freedom are lost when transforming to the DW formulation. In fact, instead of the standard notion of a degree of freedom per space point, which originates in the conventional Hamiltonian treatment, in the present multiparameter Hamiltonian description it is the (finite) number of the components of the field, which is important. The label of the conventional field degree of freedom, the space coordinate $\mathbf{x}$, goes over to the space-time multiparameter $x^{\mu} = (\mathbf{x}, t)$; that is the usual "infinite-dimensionality" of field theory in the present formulation is equivalently accounted for in the form of "multiparametricity." The same also applies to field quantization based on DW theory, which is described later.

Note that there exists an analogue of the Hamilton–Jacobi theory corresponding to the DW Hamiltonian equations (1.1). The DW Hamilton–Jacobi equation (De Donder, 1935; Kastryp, 1983; Rund, 1966; von Rieth, 1984; Weyl, 1935) is formulated for $n(n =$ the number of space–time dimensions) functions $S^{\mu} = S^{\mu}(y^a, x^{\mu})$:

$$\partial_{\mu} S^{\mu} + H\left(x^{\mu},\ y^a,\ p_a^{\mu} = \partial S^{\mu}/\partial y^a\right) = 0. \tag{1.2}$$

It naturally leads to the question as to which formulation of quantum field theory could yield this field theoretic Hamilton–Jacobi equation in the classical limit. The scheme of field quantization that is outlined in section 2.2 is a possible answer to the question and underlies the present approach to quantization of gravity.

It should be mentioned that the DW formulation is a particular case of more general *Lepagean* (Lepage, 1941, 1942) Hamiltonian-like theories for fields (known in the calculus of variations of multiple integrals (Giaquinta, 1995)), which differ by the definitions of polymomenta and the analogues of Hamilton's canonical function H (both following essentially from different choices of the Lepagean equivalents of the Poincaré–Cartan form; for further details and references see (Dedecker, 1977; De Donder, 1935; Gotay, 1991; Kastrup, 1983; Krupka, 1983, 1986, 1987; Rund, 1966; Weyl, 1935). All theories of this type treat space and time variables on an equal footing and are finite-dimensional in the sense that the corresponding analogues of the configuration and the polymomentum phase space are finite-dimensional. They all reduce to the Hamiltonian formalism of mechanics at $n = 1$.

Moreover, all these formulations are, in a sense, intermediate between the Lagrangian formulation and the canonical Hamiltonian formulation: they still keep space–time variables indistinguishable but already possess the essential features of the Hamiltonian-like description, being based on the first-order form of the field equations and a Legendre transform. Besides, there are intimate relations, not fully studied as yet, between the structures of the canonical Hamiltonian formalism and the structures of the Lepagean formulations (Gotay, 1991a,b; Gotay *et al.*, 1998; Gotay, *et al.*, in preparation; Helein and Kouneiher, 2000; Kijowski and Tulczyjew, 1979; Śniatycki, 1984) that point to the latter as a natural intermediate step when formulating the field theories canonically proceeding from the Lagrangian level. For this reason, henceforth we refer to the finite dimensional covariant Hamiltonia-like formulations based on different Lepagean theories as "*precanonical*." Further justification of the term can be found in section 4. The term "*precanonical quantization*" to be used throughout means, in most general sense, a quantization based on the Hamiltonian-like structures of a Lepagean theory. In this paper, however, we deal only with a particular Lepagean theory: the DW formulation and the corresponding quantization. Thus the term precanonical quantization will be used rather in this limited sense.

Let us note also that precanonical formulations typically have different regularity conditions than the canonical Hamiltonian formalism. For example, the DW formulation (1.1) requires that $\det \|\partial^2 L/\partial_\mu y^a \partial_\nu y^b\| \neq 0$. This condition is obviously different from the regularity condition of the canonical formalism: $\det \|\partial^2 L/\partial_t y^a \partial_t y^b\| \neq 0$. As a result, the "constraints," understood as obstacles to the corresponding generalized Legendre transforms $\partial_\mu y^a \rightarrow p_a^\mu$, have a quite different structure from that of the standard canonical formalism. In fact, the singular theories from the point of view of the canonical formalism can be regular from the precanonical point of view (as e.g., the Nambu–Goto string (Kanatchikov, 1998a)) or vice versa (as e.g., the Dirac spinor field (von Rieth, 1984)). This opens a yet unexplored possibility of avoiding the constraints analysis when quantizing within the precanonical framework by choosing for a given theory an appropriate non-singular Lepagean Legendre transformation. In fact, this possibility is exploited later, in section 3.2, when quantizing general relativity without any mention of constraints.

The idea of using the DW Hamiltonian formulation for field quantization dates back to Born (1934) and Weyl (1934). However it has not received much attention since then (see, however, Good, 1994, 1995; Günther, 1987a; Navarro, 1995, 2000; Sardanashvily, 1994). Obviously, one of the reasons is that quantization needs more than just an existence of a Hamiltonian-like formulation of the field equations: additional structures, such as the Poisson bracket (for canonical or deformation quantization), the symplectic structure (for geometric quantization), and a Poisson bracket formulation of the field equations (in order to formulate or postulate the quantum dynamical law) are necessary.

Unfortunately, in spite of a number of earlier attempts (Edelen, 1961; Good, 1954; Hermann, 1970; Marsden *et al.*, 1986; Tapia, 1988) and the progress in understanding the relevant aspects of the geometry of classical field theory, such as those related to the notion of the (Hamilton–)Poincaré–Cartan (or multisymplectic) form (Cariñena *et al.*, 1991; Goldschmidt and Sternberg, 1973; Gotay *et al.*, 1998) and Günther's polysymplectic form (Günther, 1987b), a construction which could be suitable as a starting point of quantization has been lacking. It is only recently that a proper Poisson bracket operation, which is defined on differential forms representing the dynamical variables and leads to a Poisson–Gerstenhaber algerba structure, has been found within the DW theory by Kanatchikov (1993, 1995, 1997a, 1998a) (see also Helein and Kouneiher, 2000; Forger and Römer, in press; Paufler, 2001a,b; for recent generalizations). This progress has been accompanied and followed by further developments in “multisymplectic” generalizations of the symplectic geometry aimed at applications in field theory and the calculus of variations (Betounes, 1984, 1987; Cantrijn *et al.*, 1996, 1999; Cartin, 1997; de Léon *et al.*, 1995, 1996, 2000; Edelen and Snyman, 1986; Fulp *et al.*, 1996; Giachetta *et al.*, 1997; Lawson, 1997; Olver, 1993) and in other geometric aspects of the Lagrangian and Hamiltonian formalism in field theory (Echeverría-Enríquez and Muñoz-Lecanda, 1992; Echeverría-Enríquez *et al.*, 1996, 1997, 1999; Hrabak, 1999a,b; Ibort *et al.*, 1998, 2000; Kanatchikov, 2000a), which to a great extent are been so far basically ignored by the wider mathematical physics community.

The elements of field quantization based on the aforementioned Poisson–Gerstenhaber brackets on differential forms have been discussed by Kanatchikov (1995, 1998b,c, 1999a) and will be briefly summarized in section 2.2. Unfortunately, many fundamental aspects of the corresponding *precanonical* approach to field quantization, as we suggest to call it, so far remain poorly understood (see Kanatchikov, 1998b,c and section 4 for a discusion) and require a further analysis. This particularly concerns an interplay with the standard formalism and notions of quantum field theory (see Kanatchikov, 2000b for a recent progress). Nevertheless, the already elaborated part of the theory points to intriguing features and as yet unexplored potential that, hopefully, are capable to make the precanonical approach a useful complement to the presently available concepts and techniques of quantum field theory.

The purpose of this paper is to apply the precanonical approach to field quantization, as we understand it now, to the problem of quantization of general relativity (see Kanatchikov, 1998d, 1999b, 2000c, in press; for earlier reports). We hope that this application can shed new light on the problems of quantum gravity and can be useful also for better understanding of the precanonical approach itself.

We proceed as follows: first, in section 2, we summarize basic elements of precanonical formalism and quantization based on the DW theory and then, in section 3, apply this framework to general relativity. Discussion and concluding remarks are presented in section 4.

2. PRECANONICAL FORMALISM AND QUANTIZATION BASED ON DW THEORY

In this section we briefly summarize basic elements of precanonical formalism based on the DW theory and then outline the corresponding precanonical field quantization scheme.

2.1. Classical Theory

The mathematical structures underlying the DW form of the field equations, Eq. (1.1), have been studied in our previous papers (Kanatchikov, 1997a,b, 1998a) to which we refer for more details.

The analogue of the Poisson bracket in the DW formulation is deduced from the object, called the *polysymplectic form*, which in local coordinates can be written in the form[5]

$$\Omega = -dy^a \wedge dp_a^\mu \wedge \omega_\mu$$

and is viewed as a field theoretic generalization of the symplectic form within the DW formulation. Note that if Σ, Σ: $(y^a = y^a(\mathbf{x}), t = t)$, denotes the Cauchy data surface in the covariant configuration space (y^a, x^μ), the standard symplectic form in field theory, ω_s, can be expressed as the integral over Σ of the restriction of Ω to Σ, $\Omega|_\Sigma$ (Gotay, 1991a,b; Gotay *et al.*, 1998; Gotay, in press; Śniatycki, 1984) that is,

$$\omega_S = \int_\Sigma \Omega|_\Sigma .$$

The polysymplectic form Ω associates horizontal p-forms $\overset{p}{F}$, $\overset{p}{F} := \frac{1}{p!} F_{\mu_1 \ldots \mu_p}(z^M)\, dx^{\mu_1 \cdots \mu_p} (p = 0, 1, \ldots, n)$, with $(n - p)$-multivectors (or more general algebraic operators of degree $-(n - p)$ on the exterior algebra), $\overset{n-p}{X}$, by the relation:

$$\overset{n-p}{X} \lrcorner\, \Omega = d\overset{p}{F}, \tag{2.1}$$

where $\lrcorner$ denotes the contraction of a multivector with a form. Then the graded Poisson brackets of horizontal forms representing the dynamical variables is given by

$$[\![\overset{p}{F}_1, \overset{q}{F}_2]\!] := (-)^{n-p} \overset{n-p}{X}_1 \lrcorner\, d\overset{q}{F}_2 . \tag{2.2}$$

Hence the bracket of a p-form with a q-form is a form of degree $(p + q - n + 1)$, where n is the space–time dimension. Note that, as a consequence, the subspace of forms of degree $(n - 1)$ is closed with respect to the bracket, as well as the subspace of forms of degree 0 and $(n - 1)$.

[5] Strictly speaking this object is understood as the equivalence class of forms modulo the forms of the horizontal degree n, see (Kanatchikov, 1998a) for more details. Henceforth we denote $\omega := dx^1 \wedge \cdots \wedge dx^n$, $\omega_\mu := \partial_\mu \lrcorner\, \omega = (-1)^{\mu-1} dx^1 \wedge \cdots \widehat{dx^\mu} \cdots \wedge dx^n$, $dx^{\mu_1 \cdots \mu_p} := dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$, and $\{z^M\} := \{y^a, p_a^\mu, x^\mu\}$.

This construction leads to a hierarchy of algebraic structures that are graded generalizations of the Poisson algebra in mechanics (Kanatchikov, 1997a,b, 1998a). Specifically, on a small subspace of the so-called Hamiltonian forms (i.e., those which can be mapped by relation (2.1) to multivectors) one obtain the structure of a so-called Gerstenhaber algebra (Gerstenhaber, 1963).

Let us recall, that the latter is a triple $\mathcal{G} = (\mathcal{A}, \bullet, [\![\, ,\,]\!])$, where $\mathcal{A}$ is a graded commutative associative algebra with the product operation $\bullet$ and $[\![\, ,\,]\!]$ is a graded Lie bracket which fulfils the graded Leibniz rule with respect to the product $\bullet$, with the degree of an element a of $\mathcal{A}$ with respect to the bracket operation, $bdeg(a)$, and the degree of a with respect to the product $\bullet$, $pdeg(a)$, related as $bdeg(a) = pdeg(a) + 1$. In our case the Lie bracket is the bracket operation defined in (2.2) which is also closely related to the Schouten–Nijenhuis bracket of multivector fields (the latter is related to our bracket in the similar way as the Lie bracket of vector fields is related to the Poisson bracket). Correspondingly, the graded commutative multiplication $\bullet$ is what we call the “co-exterior product”

$$F \bullet G := *^{-1}(*F \wedge *G)$$

(* is the Hodge duality operator), with respect to which the space of Hamiltonian forms is stable (Kanatchikov, 1997a,b; Paufler, 2001a,b).

Note that more general (“non-Hamiltonian”) forms give rise to a noncommutative (in the sense of Loday’s “Leibniz algebras” (Loday, 1993)) higher-order (in the sense of a higher-order analogue of the graded Leibniz rule replacing the standard Leibniz rule in the definition) generalization of a Gerstenhaber algebra (Kanatchikov, 1997a,b).

The bracket defined in (2.2) enables us to identify the pairs of “precanonically conjugate” variables and to represent the DW Hamiltonian equations in (generalized) Poisson bracket formulation. In fact, the appropriate notion of precanonically conjugate variables in the present context is suggested by considering the brackets of horizontal forms of the kind $y^a dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$ and $p_a^\mu \partial_\mu \lrcorner\, \partial_{\mu_1} \lrcorner \ldots \partial_{\mu_q} \lrcorner\, \omega$, with $p \geq q$. In particular, in the Lie subalgebra of Hamiltonian forms of degree 0 and $(n-1)$, the nonvanishing “precanonical” brackets take the form (Kanatchikov, 1998a)

$$\begin{aligned} [\![p_a^\mu \omega_\mu, y^b]\!] &= \delta_a^b, \\ [\![p_a^\mu \omega_\mu, y^b \omega_\nu]\!] &= \delta_a^b \omega_\nu, \\ [\![p_a^\mu, y^b \omega_\nu]\!] &= \delta_a^b \delta_\nu^\mu. \end{aligned} \tag{2.3}$$

This brackets obviously reduce to the canonical Poisson bracket in mechanics, $\{p^a, q_b\} = \delta_a^b$, at $n = 1$. Hence, the pairs of variables entering the brackets (2.2) and (2.3) can be viewed as precanonically conjugate with respect to the graded Poisson bracket (2.2). Note that the brackets (2.3) do not involve any dependence on space and time variables. Therefore, they can be viewed as "equal-point" brackets, as opposite to the usual "equal-time" Poisson brackets in field theory.

By considering the brackets of precanonical variables entering (2.3) with H or $H\omega$ we can write the DW Hamiltonian field equations (1.1) in Poisson bracket formulation: for example,

$$\mathbf{d}(y^a \omega_\mu) = [\![H\omega, y^a \omega_\mu]\!] = \frac{\partial H}{\partial p_a^\mu}\omega,$$

$$\mathbf{d}(p_a^\mu \omega_\mu) = [\![H\omega, p_a^\mu \omega_\mu]\!] = \frac{\partial H}{\partial y^a}\omega, \tag{2.4}$$

where $\mathbf{d}$ is the total exterior differential such that $\mathbf{d}y = \partial_\mu y(x) dx^\mu$. The DW Hamiltonian equations written in the form (2.4) point to the fact that the types of the space–time variations, which are controlled by H, are related to the operation of the exterior differentiation. This generalizes, to the present formulation of field theory, the familiar statement in the analytical mechanics that Hamilton's canonical function generates the time evolution. Note that this observation largerly underlies our hypothesis (2.6) regarding the form of a generalized Schrödinger equation within the precanonical quantization approach (Kanatchikov, 1995, 1998b, 1999a).

2.2. Precanonical Quantization

Quantization of the Gerstenhaber algebra $\mathcal{G}$ or its previously mentioned generalizations would be a difficult mathematical problem. One may even doubt that the current notions of quantization or deformation are general enough to treat the problem (Flato, 1997). This is due to the fact that $bdeg(a) \neq pdeg(a)$, for $a \in \mathcal{G}$. It is only recently that a progress has been made along the lines of geometric quantization of the Poisson–Gerstenhaber brackets (2.2) (Kanatchikov, 2000d, in preparation a), which suggests that the difficulty can be solved by admitting the operators to be nonhomogeneous in degree, at least on the level of prequantization.

Fortunately, in physics there is usually no need to quantize the whole Poisson algebra. It is even known to be impossible, in the sense of Dirac canonical quantization, as it follows from the Groenewold–van Hove "no-go" theorem (Emch, 1984; Gotay, 1998a, 1998b). In fact, quantization of a small Heisenberg subalgebra of the canonical brackets often suffices.

Therefore, it seems reasonable, at least as the first step, to quantize a small subalgebra of graded Poisson brackets that resembles the Heisenberg subalgebra of canonical variables. A natural candidate is the subalgebra of precanonical brackets in the Lie subalgebra of Hamiltonian forms of degree 0 and $(n-1)$, Eqs. (2.3). In fact, the scheme of field quantization discussed by Kanatchikov (1995, 1998b, 1999a) is essentially based on quantization of this small subalgebra by the Dirac correspondence rule: $[\hat{A}, \hat{B}] = i\hbar \widehat{\{\![A, B]\!\}}$. It leads to the following realization of operators corresponding to the quantities involved in (2.3):

$$\begin{aligned} \widehat{p_a^\mu \omega_\mu} &= i\hbar \partial/\partial y^a, \\ \hat{p}_a^\nu &= -i\hbar\kappa\gamma^\nu \, \partial/\partial y^a, \\ \hat{\omega}_\nu &= -\kappa^{-1}\gamma_\nu, \end{aligned} \tag{2.5}$$

where γ^μ are the imaginary units of the Clifford algebra of the space–time manifold and the parameter κ of the dimension $(\text{length})^{-(n-1)}$ is required by the dimensional consistency of (2.5). An identification of κ with the ultraviolet cutoff or a fundamental length scale quantity was discussed by Kanatchikov (1998b,c). The realization (2.5) is essentially inspired by the relation between the Clifford algebra and the endomorphisms of the exterior algebra (Chevalley, 1997). A crucial assumption underlying the proof that the operators in (2.5) fulfill the commutators following from (2.3) is that the composition law of operators implies the symmetrized product of γ-matrices.

The realization (2.5) indicates that quantization of DW formulation, viewed as a multiparameter generalization of the standard Hamiltonian formulation with a single time parameter, results in a generalization of the quantum theoretic formalism in which (i) the hypercomplex (Clifford) algebra of the underlying space–time manifold replaces the algebra of the complex numbers (i.e., the Clifford algebra of $(0 + 1)$-dimensional "space–time") in quantum mechanics, and (ii) n space–time variables are treated on equal footing and generalize the one dimensional time parameter. In doing so the quantum mechanics is reproduced as a special case corresponding to $n = 1$.

This philosophy leads to the following (covariant, "multitemporal," hypercomplex) generalization of the Schrödinger equation to the precanonical framework (Kanatchikov, 1998b, 1998c, 1999a)

$$i\hbar\kappa\gamma^\mu\partial_\mu\Psi = \hat{H}\Psi, \tag{2.6}$$

where $\hat{H}$ is the operator corresponding to the DW Hamiltonian function, the constant κ appears again on dimensional grounds, and $\Psi = \Psi(y^a, x^\mu)$ is the wave function over the covariant configuration space of field and space–time variables.

Equation (2.6) gives rise to the conservation law

$$\partial_\mu \int dy \, \bar{\Psi}\gamma^\mu\Psi = 0 \tag{2.7}$$

provided $\hat{H}$ is Hermitean with respect to the scalar product $\langle\Psi, \Phi\rangle = \int dy \, \bar{\Psi}\Phi$, which is also used for calculating the expectation values of operators:

$$\langle\hat{O}\rangle(x) := \int dy \, \bar{\Psi}\hat{O}\Psi. \tag{2.8}$$

The main argument in favor of a generalized Schrödinger equation (2.6) is that it satisfies at least two important aspects of the correspondence principle (Kanatchikov, 1998c, 1999a):

(i) it leads, at least in the simplest case of scalar fields, to the DW canonical Eqs. (1.1) for the mean values of the appropriate operators (the Ehrenfest theorem), for example,

$$\begin{aligned} \partial_\mu \big\langle \hat{p}^\mu_a \big\rangle &= -\langle (\partial H / \partial y^a)^{\rm op} \rangle, \\ \partial^\mu \big\langle \big(y^a \omega_\mu\big)^{\rm op} \big\rangle &= \big\langle \big(\partial H / \partial p^\mu_a \omega_\mu\big)^{\rm op} \big\rangle, \end{aligned} \tag{2.9}$$

where $(F)^{\rm op}$ denotes the operator corresponding to the variable F, and

(ii) it reduces to the DW Hamilton–Jacobi equation (1.2) (with some additional conditions) in the classical limit.

Moreover, it was shown recently that Eq. (2.6) allows us to derive the standard functional differential Schrödinger equation, once a suitable physically motivated ansatz relating the Schrödinger wave functional and the wave function in (2.6) is constructed (Kanatchikov, 2000b).

Some details on the application of the present precanonical quantization scheme to the case of scalar fields can be found in (Kanatchikov, 1998c, 1999a, 2000b). It should be noted that a capability of Eq. (2.6) to reproduce the field equations in the classical limit, that is, an infinite dimensional Hamiltonian system in the conventional sense, implies that though the generalized Schrödinger equation (2.6) is partial differential and is formulated in terms of a finite dimensional analogue of the configuration space, no field degrees of freedom are lost in this description. Similarly to the classical level, the customary "infinite-dimensionality" goes over into a "multiparametricity." Further details on the interplay between precanonical and canonical field quantization have been discussed recently in (Kanatchikov, 2000b).

3. PRECANONICAL QUANTIZATION OF GENERAL RELATIVITY

In this section we first outline a curved space–time generalization of the precanonical quantization scheme presented in section 2.2 and then discuss its further application to quantization of general relativity. The required DW Hamiltonian formulation of general relativity is discussed in section 3.2.1. The rest of section 3.2 is devoted to the derivation of a diffemorphism covariant Dirac-like wave equation for quantum general relativity. This equation is argued to include a "bootstrap condition" that introduces an averaged self-consistent classical geometry involved in the Dirac-like wave equation ensuring, in this sense, the independence of the formulation from the choice of an arbitrary background.

3.1. Curved Space–Time Generalization

To apply the precanonical framework to general relativity we first need to extend it to curved space–time with the metric $g_{\mu\nu}(x)$. The extension of the generalized Schrödinger equation (2.6) to curved space-time is similar to that of the Dirac equation, that is,

$$i\hbar\kappa\gamma^{\mu}(x)\nabla_{\mu}\Psi = \hat{H}\Psi, \tag{3.1}$$

where $\hat{H}$ is an operator form of the DW Hamiltonian function and ∇_{μ} is the covariant derivative, $\nabla_{\mu} := \partial_{\mu} + \theta_{\mu}(x)$. We introduced x-dependent γ-matrices which fulfill

$$\gamma_{\mu}(x)\gamma_{\nu}(x) + \gamma_{\nu}(x)\gamma_{\mu}(x) = 2g_{\mu\nu}(x) \tag{3.2}$$

and can be expressed with the aid of vielbein fields $e^{A}_{\mu}(x)$, such that

$$g_{\mu\nu}(x) = e^{A}_{\mu}(x)e^{B}_{\nu}(x)\eta_{AB}, \tag{3.3}$$

and the (pseudo-)Euclidian tangent space Dirac matrices γ^{A}, $\gamma^{A}\gamma^{B} + \gamma^{B}\gamma^{A} := 2\eta^{AB}$:

$$\gamma^{\mu}(x) := e^{\mu}_{A}(x)\gamma^{A}.$$

If Ψ is a spinor wave function then ∇_{μ} is the spinor covariant derivative: $\nabla_{\mu} = \partial_{\mu} + \theta_{\mu}$, where

$$\theta_{\mu} = \frac{1}{4}\theta_{AB_{\mu}}\gamma^{AB}, \quad \gamma^{AB} := \frac{1}{2}(\gamma^{A}\gamma^{B} - \gamma^{B}\gamma^{A})$$

denotes the spin connection with the components given by the usual formula

$$\theta^{A}{}_{B_{\mu}} = e^{A}_{\alpha}e^{\nu}_{B}\Gamma^{\alpha}_{\mu\nu} - e^{\nu}_{B}\partial_{\mu}e^{A}_{\nu}. \tag{3.4}$$

For example, interacting scalar fields ϕ^{a} on a curved background are described by the Lagrangian density

$$\mathcal{L} = \sqrt{g}\left\{\frac{1}{2}\partial_{\mu}\phi^{a}\partial^{\mu}\phi_{a} - U(\phi^{a}) - \xi R\phi^{2}\right\},$$

where $g := |\det(g_{\mu\nu})|$. This gives rise to the following expressions of polymomenta and the DW Hamiltonian density

$$p^{\mu}_{a} := \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^{a})} = \sqrt{g}\partial^{\mu}\phi_{a}, \qquad \sqrt{g}H = \frac{1}{2\sqrt{g}}p^{\mu}_{a}p^{a}_{\mu} + \sqrt{g}\{U(\phi) + \xi R\phi^{2}\}$$

for which the corresponding operators can be found to take the form

$$\hat{p}^{\mu}_{a} = -i\hbar\kappa\sqrt{g}\gamma^{\mu}\frac{\partial}{\partial\phi^{a}},$$

$$\hat{H} = -\frac{\hbar^2\kappa^2}{2}\frac{\partial^2}{\partial\phi^a\partial\phi_a} + U(\phi) + \xi R\phi^2. \tag{3.5}$$

3.2. Precanonical Approach to Quantum General Relativity

In the context of general relativity, the field variables are the metric $g_{\alpha\beta}$ (or the vielbein e^μ_A) components. Hence, according to the precanonical scheme, the wave function is a function of space–time and metric (or vielbein) variables, that is, $\Psi = \Psi(x^\mu, g^{\alpha\beta})$ (or $\Psi = \Psi(x^\mu, e^\mu_A)$). To formulate an analogue of the Schrödinger equation for this wave function we need γ-matrices that fulfill

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{3.6}$$

and are related to the (pseudo-)Euclidian γ-matrices γ^A by the vielbein components: $\gamma^\mu := e^\mu_A\gamma^A$, with $g^{\mu\nu} =: e^\mu_A e^\nu_B \eta^{AB}$. Note that, as opposite to the theory on curved background, the variables e^μ_A, γ^μ, and $g^{\mu\nu}$ do not carry any dependence on space–time variables x; they are instead viewed as the fibre coordinates in the corresponding bundles over the space–time. The corresponding fields $e^\mu_A(x)$, $\gamma^\mu(x)$, and $g^{\mu\nu}(x)$ exist only as classical notions and represent the sections in these bundles.

Now, modelled after (3.1), the following (symbolic form of the) generalized Schrödinger equation for the wave function of quantized gravity can be put forward

$$i\hbar\kappa\widehat{e\nabla}\Psi = \hat{\mathcal{H}}\Psi, \tag{3.7}$$

where $\hat{\mathcal{H}} := \widehat{eH}$ is the operator form of the DW Hamiltonian density of gravity, an explicit form of which is to be constructed, $e := |\det(e^A_\mu)|$, and $\hat{\nabla}$denotes the quantized Dirac operator in the sense that the corresponding connection coefficients are replaced by appropriate differential operators (cf., e.g., Eqs. (3.15), (3.16), and (3.21)). Note also, that in the context of quantum gravity it seems to be very natural to identify the parameter κ in (3.7) with the Planck scale quantity, that is, $\kappa \sim \ell_{\text{Planck}}^{-(n-1)}$.

If the wave function in (3.7) is spinor then the covariant derivative operator $\hat{\nabla}_\mu$ contains the spin connection, which on the classical level involves the term with the space–time derivatives of vielbeins (cf. Eq. (3.4)), which cannot be expressed in terms of the quantities of the metric formulation. Consequently, the spinor nature of Eq. (3.7) seems to necessitate the use of the vielbein formulation of general relativity. However, no suitable DW formulation of general relativity in vielbein variables is available so far (for a related discussion see also Esposito *et al.*, 1995; Stornaiolo and Esposito, 1997). The main problem is that the Lagrangian in vielbein formulation depends on vielbeins (4×4 components in $n = 4$ dimensions) and the spin connection (4×6 components), which involves only the antisymmetrised space–time derivatives of vielbeins. Hence, the space–time derivatives of vielbeins ($4 \times 4 \times 4$ components) cannot be expressed in the desired DW Hamiltonian form $\partial_\mu e^a_\nu = \partial H/\partial \pi_a^{\nu\mu}$ (cf. Eq. (1.1)) for any definition of H and polymomenta $\pi_a^{\nu\mu}$ because the latter will be constrained to be antisymmetric in indices μ and ν. The similar problem is encountered in DW formulation of electrodynamics (Kanatchikov, 1998a) due to the irregularity of DW Legendge transform, which is a consequence of the presence of only the antisymmetrised space–time derivatives of four-potentials in the Lagrangian. The problem, however, can be avoided if one starts from a proper gauge fixed action (Kanatchikov, in preparation) (note, that this step can be interpreted also as a choice of another Lepagean equivalent of the Lagrangian). In fact, what we need here is a precanonical analog of the analysis of irregular (in the sense of DW formulation) Lagrangians and the corresponding quantization. Unfortunately, this part of the theory remains so far to a great extent undeveloped. For this reason the subsequent consideration will be based on the metric formulation which does not suffer from these problems because the Lagrangian depends on the Christoffel symbols (4×10 components) and thus enables us to express the first derivatives of the metric (4×10 components) in DW form (cf. section 3.2.1).

As we shall see, the metric formulation also enables us to discuss the basic ingredients of precanonical quantization of gravity. In fact, one can argue that the additional degrees of freedom of the vielbein gravity, as compared with the metric gravity, that is those related to the local orientations of vielbeins, are not physical: they have to be gauged away by a coordinate gauge condition which has to be imposed in the end of the quantization procedure (cf. Eq. (3.24)). This makes the analysis based on the metric formulation even more justified from the physical point of view.

3.2.1. DW Formulation of the Einstein Equations

A suitable DW-like formulation of general relativity in metric variables was presented earlier by Hořava (1991) and Krupka and Stěpánkova (1983). In this formulation the field variables are chosen to be the metric density components $h^{\alpha\beta} := \sqrt{g}g^{\alpha\beta}$ and the polymomenta, $Q^\alpha_{\beta\gamma}$, are found to be represented by the following combination of the Christoffel symbols

$$Q^\alpha_{\beta\gamma} := \frac{1}{8\pi G}\left(\delta^\alpha_{(\beta}\Gamma^\delta_{\gamma)\delta} - \Gamma^\alpha_{\beta\gamma}\right). \tag{3.8}$$

Respectively, the DW Hamiltonian density $\mathcal{H} := \sqrt{g}H$ assumes the form

$$\mathcal{H}\big(h^{\alpha\beta}, Q^\alpha_{\beta\gamma}\big) := 8\pi G h^{\alpha\gamma}\left(Q^\delta_{\alpha\beta}Q^\beta_{\gamma\delta} + \frac{1}{1-n}Q^\beta_{\alpha\beta}Q^\delta_{\gamma\delta}\right) + (n-2)\Lambda\sqrt{g} \tag{3.9}$$

which is essentially the truncated Lagrangian density of general relativity (with the opposite sign of the cosmological term) written in terms of variables $h^{\alpha\beta}$ and $Q^\alpha_{\beta\gamma}$.

Using these variables the Einstein field equations are formulated in DW Hamiltonian form as follows

$$\partial_\alpha h^{\beta\gamma} = \partial \mathcal{H} \,/\, \partial Q^\alpha_{\beta\gamma}, \tag{3.10}$$

$$\partial_a Q^\alpha_{\beta\gamma} = -\partial\mathcal{H}/\partial h^{\beta\gamma}, \tag{3.11}$$

where Eq. (3.10) is equivalent to the standard expression of the Christoffel symbols in terms of the metric and Eq. (3.11) yields the vacuum Einstein equations in terms of the Christoffel symbols.

The present DW formulation originally was obtained by Hořava (1991) and Krupka and Štěpánkova (1983), using the theory of Lepagean equivalents. However, it can be derived also by straightforwardly applying the transformations leading to Eqs. (1.1) to the Einstein truncated Lagrangian density:

$$\mathcal{L}_{\mathrm{E}} = \frac{1}{16\pi G} h^{\mu\nu}\left(\Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\sigma\lambda}\right).$$

3.2.2. Naive Precanonical Quantization

Now, let us formally follow the curved space–time version of precanonical quantization scheme and apply it to the previously mentioned DW formulation of general relativity. This leads to the following operator form of polymomenta $Q^\alpha_{\beta\gamma}$

$$\hat{Q}^\alpha_{\beta\gamma} = -i\hbar\kappa\gamma^\alpha \left\{ \sqrt{g}\frac{\partial}{\partial h^{\beta\gamma}} \right\}_{\mathrm{ord}} \tag{3.12}$$

which is given up to an ordering ambiguity in the expression inside the curly brackets $\{\ldots\}_{\mathrm{ord}}$. By substituting this expression to (3.9) and performing a formal calculation using the assumption of the "standard" ordering (that the differential operators are all collected to the right) and relation (3.6) for curved γ-matrices we obtain the operator form of the DW Hamiltonian density, also up to an ordering ambiguity:

$$\hat{H} = -8\pi G\hbar^2\kappa^2 \frac{n-2}{n-1} \left\{ \sqrt{g} h^{\alpha\gamma} h^{\beta\delta} \frac{\partial}{\partial h^{\alpha\beta}} \frac{\partial}{\partial h^{\gamma\delta}} \right\}_{\mathrm{ord}} + (n-2)\Lambda\sqrt{g}, \tag{3.13}$$

where $\sqrt{g}$ can be obviously expressed in terms of our field variables $h^{\alpha\beta}$.

However, it should be pointed out that this procedure of the construction of operators is rather of heuristic and formal nature. In fact, according to (3.8) classical polymomenta $Q^\alpha_{\beta\gamma}$ transform as the connection coefficients while the operator associated with them in (3.12) is a tensor. Moreover, the classical DW Hamiltonian density (3.9) is an affine scalar density, while the operator constructed in (3.13) is a diffeomorphism scalar density. Therefore, we must clarify whether or not, or in which sense, this procedure is meaningful.

In the next section we shall argue how expressions (3.12) and (3.13) are valid locally, that is, in a vicinity of a point, while the information as to how to go from one space-time point to another, that is the structure of the connection, is given by the Schrödinger equation. This is very much along the lines of the precanonical approach to field quantization, which can be viewed also as the "ultra-Schrödinger" picture, in which the space–time dependence is totally transfered from operators to the wave function.

3.2.3. Covariant Schrödinger Equation for Quantized Gravity and the "Bootstrap Condition"

In order to understand the meaning of the specific realization of operators in section 3.2.2, let us remind first that the prescriptions of canonical quantization are actually applicable only in a specific coordinate system and in principle require a subsequent "covariantization." Second, let us note that the consistency of the expressions (3.12) and (3.13) with the classical transformation laws could be achieved by adding an auxiliary term in (3.12), which transforms as a connection. Then, the expression of the Christoffel symbols in terms of the polymomenta $Q^{\alpha}_{\beta\gamma}$ (cf. Eq. (3.8))

$$\Gamma^{\alpha}_{\beta\gamma} = 8\pi G \left(\frac{2}{n-1} \delta^{\alpha}_{(\beta} Q^{\delta}_{\gamma)\delta} - Q^{\alpha}_{\beta\gamma} \right) \tag{3.14}$$

would yield an operator form of the Christoffel symbols

$$\hat{\Gamma}^{\alpha}_{\beta\gamma} = -8\pi i G\hbar\kappa \left\{ \sqrt{g} \left(\frac{2}{n-1} \delta^{\alpha}_{(\beta} \gamma^{\sigma} \frac{\partial}{\partial h^{\gamma)\sigma}} - \gamma^{\alpha} \frac{\partial}{\partial h^{\beta\gamma}} \right) \right\}_{\text{ord}} + \tilde{\Gamma}^{\alpha}_{\beta\gamma}(x), \tag{3.15}$$

where the auxiliary (reference) connection $\tilde{\Gamma}^{\alpha}_{\beta\gamma}(x)$ is introduced. However, it is obvious that no arbitrary quantitity like $\tilde{\Gamma}^{\alpha}_{\beta\gamma}(x)$ should be present in a desired background independent formulation.

On the other hand, we can notice that our precanonically quantized operators arise essentially from the "equal-point" commutation relations (cf. Eqs. (2.3)) and thus can be viewed as locally defined "in a point." In an infinitesimal vicinity of a point x we always can chose a local reference system in which the auxiliary connection $\tilde{\Gamma}^{\alpha}_{\beta\gamma}(x)$ vanishes. Then one can assume that this is the reference system in which the expression (3.12) for operators $\hat{Q}^{\alpha}_{\beta\gamma}$ is valid. However, when consistently implemented, this idea requires a subsequent "patching together" procedure in order to specify how and in which sense the operators determined in different points are related to each other. This procedure is likely to lead to extra terms in our generalized Schrödinger equation (3.7) because of the connection involved in the "patching together." In fact, in accord with the essense of the "ultra-Schrödinger" picture adopted here, when all the space–time dependence is transfered from operators to the wave function, it is natural to assume that the information about the "patching together," that is, about passing from one space–time point into another, is actually controlled by the wave function and the Schrödinger equation it fulfils.

This idea can be implemented as follows. At first we formulate a generalized Schrödinger equation (3.7) in the local coordinate system in the vicinity of a point x in which the reference connection vanishes: $\tilde{\Gamma}^{\alpha}_{\beta\gamma}|_x = 0$, and then covariantize the resulting equation in the simplest way. The first step leads to a locally valid equation (cf. (3.7))

$$i\hbar\kappa\sqrt{g}\gamma^{\mu}(\partial_{\mu} + \hat{\theta}_{\mu})\Psi = \hat{\mathcal{H}}\Psi, \tag{3.16}$$

where the local operator form of the coefficients of the spin connection $\hat{\theta}_{\mu}$ (in the vicinity of x), as it follows from (3.4) and (3.15), is given by

$$\begin{aligned}\hat{\theta}^{A}_{B\mu} &= -8\pi i G\hbar\kappa\left\{e^{A}_{\alpha}e^{\nu}_{B}\sqrt{g}\left(\frac{2}{n-1}\delta^{\alpha}_{(\mu}\gamma^{\sigma}\frac{\partial}{\partial h^{\nu)\sigma}} - \gamma^{\alpha}\frac{\partial}{\partial h^{\mu\nu}}\right)\right\}_{\mathrm{ord}} + \tilde{\theta}^{A}_{B\mu}\Big|_x \\ &=: \left(\theta^{A}_{B\mu}\right)^{\mathrm{op}} + \tilde{\theta}^{A}_{B\mu}\Big|_x,\end{aligned} \tag{3.17}$$

where $(\theta^{A}_{B_\mu})^{\mathrm{op}}$ denotes the first (ordering dependent) operator term and $\tilde{\theta}^{A}_{B_\mu}|_x$ denotes a reference spin connection which ensures the correct transformation law of (3.17). Note that in general $\tilde{\theta}^{A}_{B_\mu}|_x \neq 0$ even if $\tilde{\Gamma}^{\alpha}_{\beta\gamma}|_x = 0$.

Now, in order to formulate a generally covariant version of (3.16) we notice that vielbeins do not enter the DW Hamiltonian formulation of general relativity on which the quantization in question is based. Therefore, within the present consideration they may (and can only) be treated as nonquantized classical x-dependent quantities: $e^{\mu}_{A} = \tilde{e}^{\mu}_{A}(x)$. On the other hand, the bilinear combination of vielbeins $e^{\mu}_{A}e^{\nu}_{B}\eta^{AB}$ is the metric tensor $g^{\mu\nu}$ which *is* a variable quantized (in the "ultra-Schrödinger" picture used here) as an x-independent quantity.

Both aspects can be reconciled in agreement with the correspondence principle by requiring the bilinear combination of vielbeins to be consistent with the mean value of the metric, $\langle g^{\mu\nu}\rangle(x)$, that is,

$$\tilde{e}^{\mu}_{A}(x)\tilde{e}^{\nu}_{B}(x)\eta^{AB} = \langle g^{\mu\nu}\rangle(x), \tag{3.18}$$

where the latter is given by averaging over the space of the metric components by means of the wave function $\Psi(g^{\mu\nu}, x^{\mu})$ (cf. Eq. (2.8)):

$$\langle g^{\mu\nu}\rangle(x) = \int [dg^{\alpha\beta}]\overline{\Psi}(g,x)g^{\mu\nu}\Psi(g,x), \tag{3.19}$$

with the invariant integration measure given by (cf. Misner, 1957)[6]

$$[dg^{\alpha\beta}] = \sqrt{g}^{(n+1)} \prod_{\alpha\leq\beta} dg^{\alpha\beta}. \tag{3.20}$$

Hence, the vielbein field $\tilde{e}^\mu_A(x)$ is set to be determined by the consistency with the averaged metric field. In doing so the local orientation of vielbeins is still arbitrary but it can be fixed by a proper coordinate (gauge) condition on an average vielbein field $\tilde{e}^\mu_A(x)$. A natural idea is to use the averaged vielbein field $\tilde{e}^\mu_A(x)$ to specify the quantities in the covariantized version of (3.16), like the reference spin connection, for which no operator expression can be found within the metric formulation.

Now, a diffeomorphism covariant version of (3.16) can be written in the form

$$i\hbar\kappa\tilde{e}\tilde{e}^\mu_A(x)\gamma^A(\partial_\mu + \tilde{\theta}_\mu(x))\Psi + i\hbar\kappa(\sqrt{g}\gamma^\mu\theta_\mu)^{\mathrm{op}}\Psi = \hat{\mathcal{H}}\Psi \tag{3.21}$$

which involves the self-consistent average vielbein field $\tilde{e}^\mu_A(x)$ given by the "bootstrap condition" (3.18), (3.19) and the corresponding spin connection $\tilde{\theta}_\mu(x)$. This makes the equation essentially nonlinear and integro-differential. However, the corresponding "nonlocality" is totally confined to the inner space of the metric components, over which the integration is implied in (3.19), and, therefore, does not alter the locally causal character of the equation in (a self-consistent, averaged) space–time. At the same time the nonlinearity in the left-hand side of (3.21) specifies the averaged space–time described by the tilded quantities and does not alter the quantum dynamics in the inner space, which is governed by the linear operator $\hat{\mathcal{H}}$, Eq. (3.13), and, therefore, is consistent with the superposition principle. Moreover, inasmuch as the tilded quantities present in Eq. (3.21) are introduced as resulting from the quantum averaging self-consistent with the underlying quantum dynamics of the wave function, they represent not an arbitrary a priori fixed classical background but an averaged self-consistent space–time geometry which enables us to formulate the Dirac-like equation for the wave function, as it is characteristic of the precanonical quantization approach. In this sense the formulation can be viewed as background independent.

The explicit form of the operator part of the spinor connection term in (3.21), $(\sqrt{g}\gamma^\mu\theta_\mu)^{\mathrm{op}}$, can be derived from (3.17). By assuming the "standard" ordering of operators in an intermediate calculation and replacing, when appropriate, the bilinear combinations of vielbeins appearing therewith with the metric tensor, we obtain

$$\left(\sqrt{g}\gamma^\mu\theta_\mu\right)^{\mathrm{op}} = -n\pi i G\hbar\kappa\left\{\sqrt{g}h^{\mu\nu}\frac{\partial}{\partial h^{\mu\nu}}\right\}_{\mathrm{ord}}. \tag{3.22}$$

[6] To avoid a possible confusion let us notice that the scalar product in (3.19) and the finite dimensional diiffeomorphism invariant integration measure (3.20) are mathematically well defined, in contrast to their infinite dimensional counterparts in quantum geometrodynamics based on the Wheeler–DeWitt equation.

The x-dependent reference spin connection term $\tilde{\theta}_\mu(x)$ in (3.21) is related to the average self-consistent vielbein field $\tilde{e}^A_\mu(x)$, given by the "bootstrap condition" (3.18), (3.19), and a proper coordinate condition, by the classical expression

$$\tilde{\theta}^{AB}_\mu(x) = \tilde{e}^{\alpha[A}\big(2\partial_{[\mu}\tilde{e}^{B]}{}_{\alpha]} + \tilde{e}^{B]\beta}\tilde{e}^C_\mu\partial_\beta\tilde{e}_{C\alpha}\big) \tag{3.23}$$

which is equivalent to (3.4).

Lastly, let us note that in order to distinguish a physically relevant information in (3.21) we need to impose a gauge-type condition on Ψ. The meaning of this condition is to single out a specific wave function $\Psi(h^{\mu\nu}, x^\alpha)$ from the class of wave functions which lead to the averaged metric fields which are "physically equivalent." For example, if one is to impose the De Donder–Fock harmonic gauge on the averaged metric field then the correspoding condition on the wave function reads:

$$\partial_\mu(\langle h^{\mu\nu}\rangle(x)) = 0, \tag{3.24}$$

where $\langle h^{\mu\nu}\rangle(x)$ is given similarly to (3.19) and (3.20).

Thus, we conclude that within precanonical quantization based on DW formulation, the quantized gravity is described by a generally covariant generalized Schrödinger equation (3.21), with the operators $\widehat{\mathcal{H}}$ and $(\sqrt{g}\gamma^\mu\theta_\mu)^{\mathrm{op}}$ given respectively by (3.13) and (3.22), and the supplementary "bootstrap condition" (3.18) which specifies the tilded quantities representing the self-consistent average space–time geometry.

The solutions of these equations, $\Psi(g^{\mu\nu}, x^\alpha)$, can be interpreted as the probability amplitudes of finding the values of the components of the metric tensor in the interval $[g^{\mu\nu} - (g^{\mu\nu} + dg^{\mu\nu})]$ in an infinitesimal vicinity of the point x^α. Obviously, this description is very different from the conventional quantum field theoretical one and its physical significance remains to be explored. It is interesting to note, however, that it opens an intriguing possibility to approximate the "wave function of the Universe" by the fundamental solution of Eq. (3.21). This solution is expected to describe an expansion of the wave function from the primary "probability lump" of the Planck scale and assigns a meaning to the "genesis of the space–time" in the sense that the observation of the space–time points beyond the primary "lump" becomes more and more probable with the spreading of the wave function. The self-consistency encoded in the "bootstrap condition" obviously plays a crucial role in this process: in a sense, the wave function itself determines, or "lays down," the space–time geometry it is to propagate on.

4. CONCLUDING REMARKS

The problem of quantization of gravity has been treated here from the point of view of precanonical quantization based on the structures of the De Donder–Weyl theory viewed as a manifestly covariant generalization of the Hamiltonian formulation from mechanics to field theory.

The De Donder–Weyl Hamiltonian formulation is an attractive starting point for quantization of gravity as it does not distinguish between space and time dimensions and represents the fields essentially as systems *varying* in space–time rather than as infinite dimensional systems *evolving* in time. The De Donder–Weyl Hamiltonian equations (1.1), which are equivalent to the Euler–Lagrange equations, provide us with the Hamiltonian-like description of this type of varying in space–time. These equations are formulated using the finite dimensional analogues of the confuguraion space—the space of field and space–time variables, and the phase space—the space of field and space–time variables and polymomenta.

The quantum counterpart of the theory is formulated also on a finite dimensional configuration space of field and space–time variables. The corresponding wave function $\Psi(x^\mu, y^a)$ is naturally interpreted as the probability amplitute of a field to take a value in the interval $[y - (y + dy)]$ in the vicinity of the space–time point x. In doing so all the dependence on a space–time location is transfered from operators to the wave function, corresponding to what we called the "ultra-Schrödinger" picture.

It should be noted that despite the finite dimensionality of the constructions of the precanonical approach no field degrees of freedom, understood in the conventional sense, are ignored. This is evident, on the classical level, from the fact that the DW Hamiltonian equations are equivalent to the field equations and, on the quantum level, from the observation that our generalized Schrödinger equation, Eq. (2.6), reproduces the field equations in the classical limit and also can be related to the standard functional differential Schrödinger equation (Kanatchikov, 2000b).

It is clear that the foundations of the present approach to field quantization are very different from those of the standard quantum field theory. Because of this conceptual distance it is not easy to establish a connection between both. Unfortunately, a poor understanding of this issue so far has been hindering specific applications of the approach (see, however, (Castro, 1998) for a recent attempt to apply it to quantization of p-branes).

Nevertheless, the already understood character of connections between the De Donder–Weyl theory and the standard Hamiltonian formalism (Gotay, 1991a,b; Goldschmidt and Sternberg, 1973; Sniatycki, 1984; Kijowski and Tulczyjew, 1979; Paufler and Römer, 2001) seems to provide us with a clue to a possible approach to this problem. In fact, the standard symplectic form and the standard equal-time canonical brackets in field theory can be obtained by integrating the polysymplectic form Ω and the canonical brackets (2.3) over the Cauchy data surface Σ: $(y^a = y^a(\mathbf{x}), t = \text{const.})$ in the covariant configuration space (y^a, x^μ) (Kanatchikov, 1998a). Similarly, the standard functional differential field theoretic Hamilton and Hamilton–Jacobi equations can be deduced from the partial differential DW Hamiltonian and the DW Hamilton–Jacobi equations by restricting the quantities of the DW formulation to a Cauchy data surface Σ and then integrating over it. It is natural to expect that a similar connection can be established between the elements of the precanonical approach to field quantization and those of the standard canonical quantization.

A related way to find a connection with the conventional quantum field theory is to view the Schrödinger wave functional $\Psi([y(\mathbf{x})], t)$ (see, e.g., Hatfield, 1992) as a composition of amplitudes given by our wave function $\Psi(y, \mathbf{x}, t)$ restricted to Σ. In fact, developing an earlier demonstration of this connection in the ultralocal approximation (Kanatchikov, 1998b,c) we have shown recently (Kanatchikov, 2000b) that the Schrödinger wave functional can be represented as the trace of the positive frequency part of the continual product over all spacial points of the values of the wave function $\Psi(y^a, x^\mu)$ restricted to a Cauchy surface. Besides, it has been shown that using this ansatz the standard functional differential Schrödinger equation can be derived from our Dirac-like generalized Schrödinger equation, Eq. (2.6). It is natural to ask if this kind of interplay between precanonical and canonical quantization could be extended to gravity in order to understand a possible relation between the Dirac-like wave equation for quantized gravity proposed in section 3.2.3 and the Wheeler–DeWitt equation.

The character of the interrelations, as outlined earlier, between the DW formulation (and more general Lepagean theories Dedecker, 1977; Krupka, 1983, 1986, 1987; Lepage, 1941, 1942) and the conventional canonical formalism is the reason to refer to the former as the *precanonical* formalism. The term reflects an intermediate position of the DW formulation (and its Lepagean generalizations) between the covariant Lagrangian and the "instantaneous" Hamiltonian levels of description. Note that in mechanics, $n = 1$, the precanonical description coincides with the canonical one and it is only in field theory, $n > 1$, that they become different. The same is valid for *precanonical quantization* underlying the present approach to quantization of gravity.

It should be noticed that the application of the precanonical framework to gravity immediately raises many questions to which no final answers can be given as yet. Some of these, such as, for example,

(i) how the spinor wave function is reconciled with the boson versus fermion nature of the fields we quantize;
(ii) if it can or should be replaced with a more general Clifford algebra valued wave function;
(iii) to what extent one can rely on the prescription of quantum averaging (2.8) if the underlying scalar product is neither positive definite nor x-independent;
(iv) how to quantize the operators more general than those entering the precanonical brackets (2.3); and, at last,
(v) how to calculate the observable quantities of interest in field theory using the precanonical framework,

concern rather the precanonical approach in general and are still being investigated. We hope to address them elsewhere. Let us instead concentrate here on a few questions related to the specific application of precanonical quantization to general relativity.

A severe difficulty we encountered is related to the nontensorial nature of the basic quantities (the polymomenta and the DW Hamiltonian) of the DW formulation of general relativity in section 3.2.1, which is in disagreement with the tensorial character of operators which only can be constructed (in a background independent fashion) as their quantum counterparts. The origin of this difficulty is in the fact that the DW formulation of general relativity in section 3.2.1 is based essentially on the Einstein noncovariant truncated Lagrangian density, which contains no second-order derivatives of the metric, instead of the generally covariant Einstein–Hilbert Lagrangian density $\sqrt{q}R$. To use the latter we would need a generalization of the precanonical constructions outlined in sections 2.1 and 2.2 to the second-order irregular Lagrangians (see, e.g., Aldaya, 1978; Gotay, 1991; Krupka, 1983, 1986, 1987 and the references in Gotay, 1991), which is largely not developed as yet. The vielbein formulation of general relativity in the second-order formalism would face a similar difficulty. An attempt to use the first-order (Palatini) formalism (cf. Esposito *et al.*, 1995; Gotay *et al.*, 1998; Stornaiolo and Esposito, 1997) also leads to highly irregular Lagrangians which require a proper adaptation of the precanonical treatment yet to be developed.

In the approach of this paper, the difficulty mentioned earlier is circumvented by quantizing locally, in a vicinity of a point, and then covariantizing. Though this procedure involves external elements, such as a reference vielbein field $\tilde{e}_a^\mu(x)$ and the corresponding spin connection $\tilde{\theta}_\mu(x)$ which enter into the generalized Schrödinger equation (3.21) as nonquantized quantities, those are not arbitrary since the correspondence principle requires the reference vielbein field to be consistent with the mean value of the metric. This requirement leads to a self-consistency in the theory in the form of the "bootstrap condition" (3.18) and (3.19) which connects the bilinear combination of the reference vielbein fields $\tilde{e}_a^\mu(x)$ with the quantum mean value of the metric. By this means the allowable classical geometry emerges in the theory as an approximate notion—a result of quantum averaging—self-consistent with the underlying quantum dynamics. In this sense the theory appears to be independent of an arbitrarily chosen background.

This point of view, though looking less radical than the usual denial of any background geometrical structure in quantum theory of gravity, which is known to lead to the most of the conceptual difficulties of the latter, seems to offer a working alternative to the currently more popular attempts to proceed from a postulated specific model of quantum "pregeometry" near to the Planck scale, be it a discrete space–time, a space–time foam, a noncommutative or fuzzy space–time, or the spin networks and a spin foam recently proposed within the Ashtekar program (for a review and further references see, e.g., Rovelli, 1998).

On the other hand, the appearance in the left hand side of (3.21) of, essentially, an averaged Dirac operator, may imply an approximate, "smeared down," not ultimately quantum, character of the description achieved here. In this case a further step could be required which would allow us to treat the Dirac operator in the left hand side of (3.7) beyond the framework of classical geometry. In this case a proper insight into a quantum pregeometry could be important indeed.

Let us mention also that the coefficients involving n in (3.13) and (3.22) at the present stage cannot be considered as reliably established. This is related both to the ordering ambiguity and to the unreliability of the results obtained by formal substitution of polymomenta operators (3.12) to classical expressions (for example, applying the similar procedure to the DW Hamiltonian of a massless scalar field ϕ yields the operator $-\frac{n}{2}\hbar^2\kappa^2\partial^2_{\phi\phi}$ instead of the correct one $-\frac{1}{2}\hbar^2\kappa^2\partial^2_{\phi\phi}$ (Kanatchikov, 1998b, 1999a). Note also, that at this stage it is also rather difficult to choose between the formulation based on the operator of DW Hamiltonian $\hat{H}$ and that based on the corresponding density $\hat{\mathcal{H}}$. In the former case, the generalized Schrödinger equation, Eq. (3.7), would be modified as follows: $i\hbar\kappa\widehat{\gamma^\mu\nabla_\mu}\Psi = \hat{H}\Psi$, which in general is different from (3.7) because of the ordering ambiguity. A preliminary consideration of the toy one-dimensional models corresponding to the formulations using $\hat{H}$ and $\hat{\mathcal{H}}$ respectively indicates (Kanatchikov, in preparation) that the latter formulation, which leads to a toy model similar to that discussed long ago by Klauder (1969, 1970, 1980), seem to reveal more interesting behavior and thus might be more suitable. However, to present more conclusive results, an additional analysis, possibly based on quantization of more general dynamical variables than those involved in the precanonical brackets (2.3), is required. Besides, as we have already pointed out, the vielbein formulation of general relativity can be more adequate to the application of precanonical quantization to gravity, though it is unlikely to be a panacea from the problems we have outlined earlier. The corresponding analysis is in progress and we hope to report on the results elsewhere.

In conclusion, let us summarize in short the potential advantages of the present approach. The obvious advantage is its manifest covariance (more precisely, the starting point and the resulting equations are covariant though some intermediate steps still are not). This allows us to avoid the usual restriction to globally hyperbolic space–times which is necessarily imposed in canonical quantum gravity. However, this advantage, though potentially important for considering the expected quantum topology and signature changes in quantum gravity, still could be viewed by a sceptic as a purely technical achievement. Another technical advantage is that the analogue of the Schrödinger equation and other elements of the formalism reveal no problems with their mathematical definition (the ordering problem encountered here is, in fact, not more complicated than that in quantum mechanics), in contrast to the approaches based on the Wheeler–DeWitt equation or the path integral. This advantage, however, is in-built in the precanonical approach itself, which avoids treating fields as infinite-dimensional systems, and is not specific to the quantum gravity.

As far as the physical aspects of the theory are concerned, an intriguing feature of the approach is the appearance of a self-consistently incorporated averaged vielbein field in the generalized Schrödinger equation (3.21). This enables us to avoid the direct tackling with the problems of quantum pregeometry, that is, an ultimate description of "quantum space–time" near to the Planck scale (for a recent discussion see Isham, 1993, 1997; Isham and Butterfield, 2000), which are usually viewed to be the central issue of quantum gravity. Nevertheless, in spite of not giving an insight as to what the quantum space–time, or pregeometry, could be, the present treatment refers to the classical space–time only as an approximate notion resulting from the quantum averaging and a self-consistency. No arbitrarily fixed classical background geometry is been involved. This essentially amounts to a background independence of the formulation.

Besides, the appearence of a self-consistent vielbein field provides us with a framework for discussing the problem of emergence of classical space–time in quantum gravity (for a recent review see, Isham and Butterfield, 1999a). Moreover, it could shed light on the problem of interpretation (or, that of an "external observer") in quantum cosmology since the generalized Schrödinger equation (3.21) essentially describes a sort of self-referential quantum system, in the sense that the self-consistent averaged vielbein field can be viewed as the vielbein field representing the macroscopic "self-observing" degrees of freedom of a quantum gravitational system.

ACKNOWLEDGMENTS

I thank J. Klauder for drawing my attention to his earlier papers (Klauder, 1969, 1980; Klauder and Aslaksen, 1970), A. Borowiec and M. Kalinowski for valuable remarks, C. Castro and J. Sławianowski for stimulating encouragement and useful comments, and M. Pietrzyk for her effective numerical analysis of a simplified 1D version of Eq. (3.21), which was very helpful for understanding its behavior. I also thank A. Wipf and the Institute of Theoretical Physics of the Friedrich Schiller University of Jena for their kind hospitality and excellent working conditions which enabled me to write this paper.

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